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The Recurrence of the Initial State in the Numerical Solution of the Vlasov Equation

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The approximate recurrence of the initial state, observed recently in the numerical solution of Vlasov's equation by a finite-difference Eulerian model, is shown to be a property of three independent numerical methods. Some of the methods have exponentially growing modes (Dawson's beaming instabilities), and some others do not. The recurrence is in fact a manifestation of the finite velocity resolution of the numerical methods—a property which is independent of the approximation of a plasma by a finite number of electron beams. The recurrence is shown explicitly in the numerical simulation of Landau damping by three different methods: Fourier–Hermite, the recent variational method of Lewis, and the Eulerian finite-difference method.

1. INTRODUCTION

An asymptotic expression for the small perturbation of an equilibrium Maxwellian electron distribution obeying Vlasov's equation was obtained by Landau [1] in the form $f(x, v, t) \sim \exp(ikx) \exp(-ikvt)$. This means that for the linearized Vlasov equation the distribution function develops asymptotically a velocity frequency kt, that grows beyond bound in time. The development of higher and higher frequencies in velocity space, often designated as the development of fine structures in the distribution, creates a fundamental difficulty for the numerical solution of Vlasov's equation which has not been surmounted. Because all numerical methods have a finite velocity resolution, it is clear that the simulations of the linear Vlasov equation will cease to be valid when the distribution reaches a

velocity frequency equal to the maximum frequency that can be represented by the numerical scheme. The aim of this paper is to show that most of the standard numerical methods fail in precisely the same qualitative way: after the upper time limit for the validity of the numerical simulation is reached, an approximate recurrence of the initial state takes place, and thereafter the computation has the same qualitative behavior as that observed after the initial time.[1] In some recent work [2–4], this recurrence of the initial state has been discussed in connection with the dispersion relation for a finite number of electron beams with a Maxwellian envelope, originally introduced by Dawson [5]. In this paper, we wish to emphasize that the recurrence is due to the finite velocity resolution of the numerical methods, and that it does not require that the methods have dispersion relations equal or similar to Dawson's. This main conclusion is proved by showing the recurrence of the initial state obtained in the simulation of Landau damping by the Fourier-Hermite method, whose dispersion equation has *no* growing modes [6]. This recurrence, observed also with a finite-difference method without growing modes, is qualitatively similar (in a sense to be made precise later) to those obtained with the methods used by Lewis [2], Denavit [3], and Brackbill [4], whose dispersion relations have growing modes because they are identical to the one obtained by Dawson for a finite number of beams [5]. Therefore, as all these methods show the recurrence of the initial state in the same qualitative way, this recurrence is independent of the existence of growing modes (beaming instabilities) or of the approximation of a plasma by a finite number of beams.

In Section 2, we review briefly the results obtained by Grant and Feix [6] for the dispersion relation of the Fourier–Hermite method, and also those obtained by Lewis [2] in the numerical analysis of Dawson's dispersion relation.

In Section 3, we give the results of the numerical experiments on Landau damping and their interpretation. The numerical experiments were carried out using the Fourier–Hermite method [6–8], and a finite-difference Eulerian method [9, 10]. This method has been used to simulate Landau damping by a plasma model without growing modes (beaming instabilities); in this case, the electric field behaves qualitatively in the same way as that obtained by the Fourier–Hermite method. The finite-difference program was then modified to simulate Landau damping by a multibeam model (a numerical method is said to approximate a plasma by a multibeam model if its dispersion relation is the same as Dawson's); in this case, the time behavior of the electric field is qualitatively the same as that obtained by Lewis [2]. In this way, we obtain a precise description of a plasma modeled by a finite number of equally-spaced beams with a Maxwellian envelope (i.e., Dawson's model), which allows us to separate clearly the recurrence of the initial

[1] As far as we know, the particle simulation method is the only existing method which does not show the recurrence of the initial state (see Ref. [3]).

state (common to all the numerical methods discussed) from the effects due to the beaming instabilities (which are characteristic only of multibeam models). This description is consistent with the numerical analysis of Dawson's dispersion equation carried out by Lewis [2]. A clear understanding of these numerical effects is essential for the proper interpretation of the computations of nonlinear large-amplitude plasma oscillations, a problem which is the subject of a forthcoming paper.

2. Dispersion Equations of Numerical Methods for Vlasov's Equation

Here we review and compare briefly the results obtained for the dispersion equation of the Fourier–Hermite method [6], and those obtained for Dawson's dispersion equation [2]; it is recalled that multibeam numerical methods [2–4] have dispersion equations identical to Dawson's. A clear understanding of these results is essential for the proper interpretation of the numerical experiments to follow.

In the Fourier–Hermite method for the solution of the linearized Vlasov equation (the notations and units of Refs. [7] and [8] are used here)

$$\frac{\partial f(x, v, t)}{\partial t} + v \frac{\partial f}{\partial x} - E(x, t) \frac{\partial f_0}{\partial v} = 0,$$

$$\frac{\partial E}{\partial x} = 1 - \int_{-\infty}^{\infty} f \, dv, \tag{1}$$

where f_0 is the equilibrium electron distribution and E the electric field, the perturbed distribution is expanded as follows:

$$f(x, v, t) \approx \sum_{n=-J}^{J} \exp(inkx) \sum_{m=0}^{M} \exp(-\tfrac{1}{2}v^2)\, h_m(v)\, Z_{mn}(t), \tag{2}$$

where h_m are the orthonormal Hermite polynomials. The unknown time-dependent coefficients $Z_{m1}(t)$ corresponding to the fundamental Fourier mode, $\exp(ikx)$, satisfy the following linear system of homogeneous differential equations [7, Eq. (A1)]:

$$d\mathbf{Z}/dt = \mathscr{A}\mathbf{Z}(t), \tag{3}$$

where the vector $\mathbf{Z}$ and the matrix $\mathscr{A}$ are given by

$$\mathbf{Z}(t) \equiv \begin{bmatrix} Z_{01}(t) \\ Z_{11}(t) \\ \vdots \\ Z_{M1}(t) \end{bmatrix},$$

$$\mathscr{A} \equiv \begin{bmatrix} 0 & k & & & & & \\ -\dfrac{1+k^2}{k} & 0 & -2^{1/2}k & & & & \\ & 2^{1/2}k & 0 & 3^{1/2}k & & & \\ & & -3^{1/2}k & 0 & -4^{1/2}k & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & & -M^{1/2}k & \\ & & & & M^{1/2}k & 0 & \end{bmatrix}, \tag{4}$$

As the system (3) is linear, its solutions are expressed in the form $\exp(i\omega t)$, where the ω's are the roots of the dispersion equation

$$|\mathscr{A} - i\omega\mathscr{I}| = 0. \tag{5}$$

Grant and Feix [6] have proved that all roots ω of the dispersion equation (5) *are real*; further, it can also be shown that all the roots occur in pairs of equal magnitude and opposite signs when $M + 1$ (the number of terms in the Hermite expansion) is even; if $M + 1$ is odd there is also one root $\omega = 0$. In other words, all the normal modes of the Fourier–Hermite method are standing modes—they do not damp or grow. The general solution of (3) is written in scalar form as follows

$$Z_{m1}(t) = \sum_{j=0}^{M} C_{mj} \exp(i\omega_j t), \qquad m = 0, 1, \ldots, M, \tag{6}$$

where the C_{mj} are $(M + 1)^2$ constant coefficients to be determined. As is known from the theory of homogeneous systems of ordinary differential equations with constant coefficients [11], $M^2 + M$ of the coefficients C_{mj} are determined by requiring that the expressions (6) satisfy the system (3); the remaining $M + 1$ arbitrary constants C_{mj} are required to satisfy the initial conditions. It is recalled here that the most important physical quantity, the electric field, is proportional to the coefficient $Z_{01}(t)$ [7, 8]. The implications of the result (6) for the proper interpretation of the numerical experiments will be discussed in the next section.

Lewis [2] has carried out a detailed study of Dawson's dispersion equation,

$$\omega_0^2 \sum_{j=1}^{N} \frac{n_j}{(\omega - kv_j)^2} = 1, \tag{7}$$

where $\omega_0 = (4\pi ne^2/m)^{1/2}$ is the plasma frequency, and n_j and v_j are the electron density and velocity of the jth beam. His results are compared now with those obtained for the dispersion equation of the Fourier–Hermite method. Dawson's dispersion equation has real roots (see Fig. 1(b) of Ref. [2]) which must occur in pairs of equal magnitude and opposite sign, as one can see by inspection of Eq. (7) assuming a Maxwellian or more generally a symmetric beam distribution; it is recalled that this result is also true for the Fourier–Hermite dispersion equation. We speculate that Dawson's and the Fourier–Hermite dispersion equations have corresponding real roots which are quite close to each other in magnitude. In addition, Dawson's dispersion equation has complex conjugate roots whose real parts have a magnitude (giving the oscillation frequency) which is much smaller than the magnitude of the largest real roots (see Fig. 1(b) of Ref. [2]). These exponentially growing modes give rise to the so-called beaming or multistream instability. It is important to stress here that, because the oscillation frequency of the growing modes is much smaller than the largest frequency of the standing modes, the growth of the beaming instabilities will be appreciable only in a time scale which is long compared with the period of the fastest oscillations.

3. Landau Damping, Recurrence, and Beaming Instabilities

The approximate recurrence of the initial state is first shown (see Fig. 1) in a computation of Landau damping by the Fourier–Hermite method [6–8], and by a finite-difference Eulerian method [10]. The wave number (in units of the reciprocal of the Debye length) is $k = 0.5$, and the number of terms kept in the Hermite expansion was 100, the same as in the paper of Grant and Feix [6]; this computation was also carried out by Armstrong [7] and by Lewis [2].

The computation of Grant and Feix (see Fig. 2 of Ref. [6]) shows the proper Landau damping up to a time $t = 20$; afterwards, a gradual deviation from the correct Landau slope is observed. They state that this gradual deviation is caused by the truncation of the Hermite expansion at 100 terms. In order to explain this, they point out that the maximum velocity frequency that can be represented by a Hermite expansion with N terms is $N^{1/2}$, because the asymptotic behavior of the Hermite polynomials for large N is $h_N(v) \sim \sin(N^{1/2}v)$; using this in conjunction with Landau's result $f(x, v, t) \sim \exp(ikx)\exp(-ikvt)$, they obtain an estimate for the upper time limit of validity of the Fourier–Hermite method, $T = N^{1/2}k^{-1}$, which for the computation shown in Fig. 1(a) gives $T = 20$. In contrast with the Grant and Feix results, our computation is in excellent agreement with Landau's theory up to $t \approx 30$, as can be seen by inspection of Fig. 1(a). Actually, the oscillation frequency and damping rate obtained from the numerical output averaged over the maxima occurring from $t = 4.73$ to 29.14 [3rd and 14th maxima in Fig.

1(a)] are 1.4155 ω_0 and 0.15334 ω_0, to be compared with the exact values obtained from Landau's dispersion equation, 1.4156 ω_0 and 0.15336 ω_0 [12]. The numerical integration of the system of equations (3) was carried out by a fourth-order Runge–Kutta method with a time step $\Delta t = 0.02$. The initial conditions used were the same as Armstrong's, i.e., $Z_{01}(0) =$ constant, $Z_{ml}(0) = 0$, $m \neq 0$, corresponding to a Maxwellian velocity distribution of the monochromatic $\exp(ikx)$ initial perturbation.

We must conclude that the gradual deviation from the correct behavior in the Grant and Feix computation during the interval $20 \leqslant t \leqslant 30$ can only be due to numerical inaccuracies. At about $t = 32$, the computation in Fig. 1(a) shows in Lewis words [2] an "explosive deviation" from the correct continuous results. This is the approximate recurrence of the initial state, and is a characteristic not only of the variational method of Lewis (see Fig. 4(a)) or of the distribution pushing method used by Brackbill and others [4, 13], but is common to most standard methods of solution of the collisionless Vlasov equation because they all have a finite velocity resolution (see, however, footnote 1). It is of interest to note here that after the recurrence, the damping of the electric field is much smaller than the initial Landau damping, but the oscillation frequency is quite close to Landau's frequency. In a previous paper where the electric field performed steady-state oscillations (see Fig. 2 of Ref. [10]), the recurrence was much more striking than here, because after the explosive deviation the steady-state oscillations were almost identical to those observed after the initial time.

The computation shown in Fig. 1(a) is proof that the explosive deviation from the correct Landau behavior has nothing to do with the beaming instability. This is because the dispersion equation for the Fourier–Hermite method [see Eqs. (5) and (6)] has only real roots, i.e., the normal modes of the Fourier–Hermite method are standing modes–there are *no* growing modes.

What is fascinating about Landau's problem is that in the Fourier–Hermite computation, the electric field (which is proportional to Z_{01}, see Eq. (6) with $m = 0$) oscillates with the proper frequency and damping rate predicted by Landau for a nondissipative system with an infinite number of degrees of freedom, the correct damping being obtained by the "phase-mixing" of the $M + 1$ real frequencies ω_j. A more complete discussion of this feature is given by Grant and Feix [6].

In Fig. 1(b) we show the recurrence of the initial state obtained in the simulation of Landau damping by a finite-difference Eulerian method [10]. The behavior of the electric field and its recurrence are similar in every respect to those observed with the Fourier–Hermite method. It should be realized that in this finite-difference computation, the equilibrium Maxwellian distribution is *not* approximated by a finite number of δ-function beams with a Maxwellian envelope; rather, at *any* point in velocity space the derivative of the distribution is defined smoothly by using finite-difference expressions obtained from the values of the distribution

given at the neighboring grid points. It is for this reason that the finite-difference computation shown in Fig. 1(b) is not subject to beaming instabilities, and is therefore similar in every respect to the Fourier–Hermite computation. For a velocity mesh width Δv, the highest velocity frequency that can be represented is $\pi/\Delta v$; using this in conjunction with Landau's result

$$f(x, v, t) \sim \exp(ikx) \exp(-ikvt),$$

we get the estimate for the upper time limit of validity of a finite-difference computation, $T = \pi/k\,\Delta v$ [9, 10]. It should be stressed that these estimates are obtained using Landau's linear result for the asymptotic electron distribution and, therefore, are strictly valid only for the computation of the electron distribution in Landau's model problem. However, the computations in Fig. 1 show that the electric fields

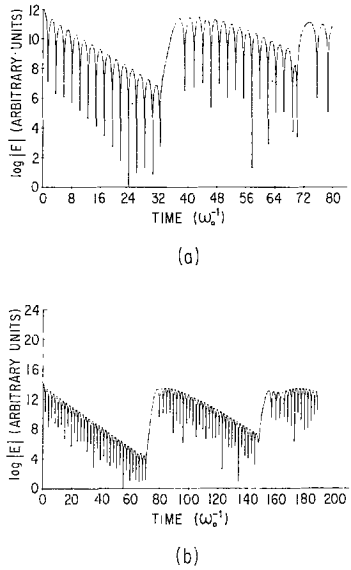

FIG. 1. The approximate recurrence of the initial state without the multibeam instability for $k = 0.5$. (a) Fourier–Hermite method with 100 terms; (b) Finite-difference method with $\Delta v = (\pi/20)\, v_t$.

are obtained correctly during a time interval longer than T; for the finite-difference computations, the superperiod (time elapsed between two successive recurrences of the electric field) is almost $2T$. This property of Vlasov's equation can be understood by inspection of Fig. 2 and 3. In Fig. 2, we plot at successive instants of time the

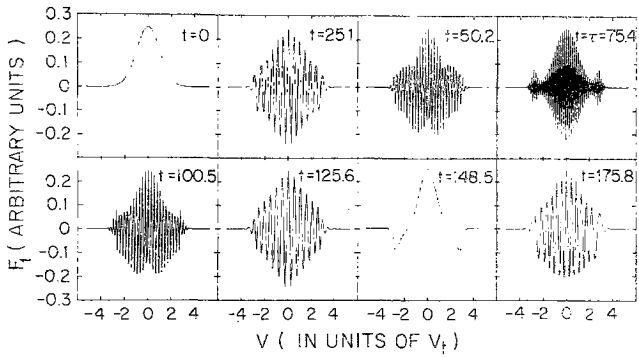

FIG. 2. Time sequence of the velocity distribution $f_1(v, t)$ defined by Eq. (8). Finite-difference method with $k = 0.5$, and $\Delta v = (1/12)\, v_t$. Time in units of ω_0^{-1}.

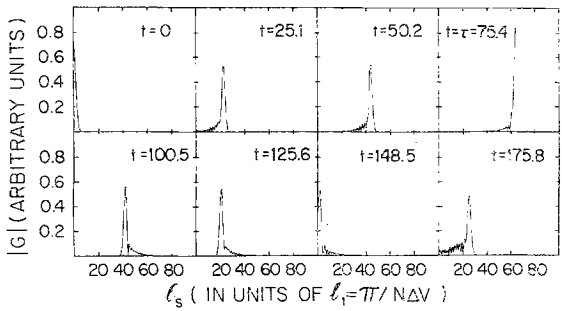

FIG. 3. Time sequence of the velocity spectrum $| G(l_s, t)$ defined by Eq. (9). Finite-difference method with $k = 0.5$, and $\Delta v = (1/12)\, v_t$. Time in units of ω_0^{-1}.

real part of the first Fourier component of the perturbed electron distribution obeying Eq. (1), i.e.,

$$f_1(v, t) = \frac{1}{L}\int_0^L f(x, v, t)\exp(-ikx)\, dx, \qquad k = \frac{2\pi}{L}, \tag{8}$$

where L is the fundamental domain length. The parameters characterizing this finite-difference computation are $k = 0.5$ and $\Delta v = (1/12)\, v_t$, where v_t is the thermal velocity. The initial perturbation $f(x, v, o)$ is spatially monochromatic $\exp(ikx)$ and has a Maxwellian distribution (see Fig. 2 for $t = 0$). As time advances, the distribution (8) develops the fine structures predicted by Landau's theory; finally, at $t \approx T = \pi/k\, \Delta v = 75.4$, the maximum fine structure (or shortest wavelength in velocity space) that can be represented by the finite-difference grid is reached, and thereafter the numerical simulation "unfolds" the distribution which becomes progressively smoother until the time $t \approx 2T = 150.8$, at which an

approximate recurrence of the initial state (the Maxwellian distribution) takes place, and the development of the fine structures starts once again.

In Fig. 3, we show the absolute value of the Fourier velocity spectrum of the time-dependent distribution given by Eq. (8), i.e.,

$$G(l_s, t) = \int_{-v_{\max}}^{v_{\max}} f_1(v, t) \exp(-il_s v)\, dv, \qquad l_s = s\frac{\pi}{v_{\max}}, \qquad s = 0,1,2,\ldots, \quad (9)$$

where $v_{\max}$ is defined by the velocity grid spacing as $v_{\max} = N\,\Delta v$, and $2N + 1$ is the number of grid points. It should be recalled [7, 8] that the electric field is at all times proportional to the amplitude of the Maxwellian part of the distribution $f_1(v, t)$, which in turn is proportional to the spectrum amplitude for $l_s = 0$. Because the initial perturbation is Maxwellian, at $t = 0$ the spectrum amplitude for $l_s = 0$ is largest. As time advances, the narrow velocity spectrum moves to the right in the manner of a travelling wave. When the spectrum reaches the maximum l_s available in the computation grid ($l_s = 63$ in Fig. 3) at $t \approx T = \pi/k\,\Delta v = 75.4$, a reflection takes place and thereafter the spectrum travels to the left, till at $t \approx 2T$ it reaches again the left "wall" ($l_s = 0$) and the approximate recurrence takes place, at which time the electric field once again reaches a value of the same order as the initial value.

It should now be perfectly clear that the sudden deviation from the correct behavior shown in Fig. 1 is *not* a numerical instability, i.e., the solution of the system of equations (3) can be obtained numerically with arbitrary accuracy during *all the time interval* shown in Fig. 1(a). Of course, Eq. (3) represent a non-dissipative system with a finite number of degrees of freedom, and cannot approximate indefinitely a nondissipative system with an infinite number of degrees of freedom (the collisionless plasma modelled by Vlasov's equation), where the damping is due to the dispersal of the electrostatic energy initially concentrated in one degree of freedom into an infinite number of degrees of freedom [14].

A. *Effects of the Multistream Instability*

The computations shown in Fig. 1 indicate that the recurrences of the initial state, taking place roughly at $2T \approx 2N^{1/2}k^{-1}$ (Fourier–Hermite method) and $2T \approx 2\pi(k\,\Delta v)^{-1}$ (finite-difference method), are only approximate. Here we are not interested in more accurate recurrences which might possibly take place after periods of time orders of magnitude higher than the above T [15]. It is now shown that, at least for a sufficiently short time, the recurrences are not prevented by the multistream instability, but that this has only a certain quantitative effect on them.

After the original theoretical discussion of Dawson on the Landau damping of a finite number of electron beams [5], Lewis has recently carried out a numerical analysis of Dawson's dispersion equation, which is also the dispersion equation

for his variational method of solution of Vlasov's equation [2]. We now describe a numerical experiment carried out with our finite-difference Eulerian method in which the plasma is approximated by a finite number of beams with a Maxwellian envelope. This is realized numerically by using the following initial perturbation in the distribution

$$f(x, n\,\Delta v, 0) = \begin{cases} \alpha \exp[-(n\,\Delta v)^2/2] \cos kx, & n = 4n_b\,, \\ 0, & n \neq 4n_b\,, \end{cases}$$
$$n_b = -N_b\,, -N_b + 1, \ldots, N_b\,, \tag{10}$$
$$n = -4N_b\,, -4N_b + 1, \ldots, 4N_b\,,$$

where Δv is the grid spacing. Thus, we simulate $2N_b + 1$ approximate δ-function beams over a velocity grid having four times as many grid points. In other words, for each grid point with an electron beam there are six grid points (three to the left and three to the right) in which the distribution is set equal to zero. The aim of this computation was to simulate Dawson's model, and therefore, to obtain a Landau-damped electric field in qualitative agreement with that observed in Lewis' simulation. The results of Lewis' computation and of our own are given in Fig. 4, and show a complete qualitative agreement with each other, as expected. Because in

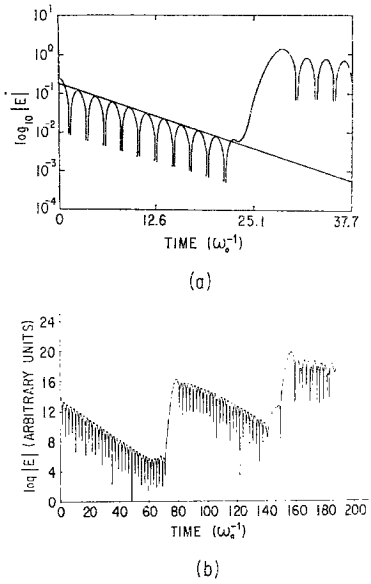

FIG. 4. The approximate recurrence of the initial state with the multibeam instability present, $k = 0.5$. (a) Variational method of Lewis [2]; (b) Finite-difference method with $\Delta v = (\pi/80)\, v_t$.

these computations the plasma is approximated by a multibeam model, their dispersion equations have exponentially growing modes. The effect of these multistream instabilities is to increase at the recurrence the magnitude of the electric field relative to the initial value. Even after the first recurrence, the multistream instability is not strong enough to prevent the damping of the electric field which results from the phase mixing of all the modes of the dispersion equation. A comparison of the computations shown in Fig. 1(b) and 4(b) indicates quite clearly that, other than the increase of the electric field at each recurrence, the qualitative effects of the multistream instability on the electric field are small. More explicitly, the "superperiod" (time elapsed between two recurrences), the electric field oscillation frequency and its damping rate are almost identical in the computations shown in Fig. 1(b) and 4(b) during the time interval $0 \leqslant t \leqslant 4T$. This is in agreement with Dawson's and Lewis' analyses, which show that the time scale for the growth of the electric field due to the multistream instability is large compared to the Landau oscillation frequency. Of course, after a time sufficiently long for several recurrences to have taken place, the multistream instability will finally dominate the damping due to the phase-mixing of the modes.

4. Conclusion

We have established that when the linear Vlasov equation is solved numerically by a method without growing modes (beaming instablities), the approximate recurrence of the initial state is related to the fact that the numerical approximation represents a non-dissipative dynamical system with a finite number of degrees of freedom. That these systems should have recurrence properties is physically reasonable, because sooner or later their characteristic standing modes might again be in almost the same phase as that corresponding to the initial condition. A precise analogue of this focusing in time of the standing modes of the truncated Vlasov's equation is the focusing in space of the solitons observed in the numerical solution of the Korteweg–de Vries equation with periodic boundary conditions, which also leads to the approximate recurrence of the initial condition [16]. It should be noted that, just as the (truncated) Fourier–Hermite approximation to Vlasov's equation, the Korteweg–de Vries equation represents also a nondissipative system where most of the energy seems to be concentrated in a finite number of degrees of freedom, i.e., the eight solitons of the Zabusky and Kruskal computation [16].

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